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Orthogonal Function Sets Arising from Integral Equations.[*]

BY O. D. KELLOGG.

1. *Introduction.*

As is well known,[†] $K(x, y)$ being a real, continuous, symmetric function, not identically zero, on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, there is at least one value of λ for which the integral equation

$$\phi(x) = \lambda \int_0^1 \phi(y) K(x, y) dy \tag{1}$$

has a solution, $\phi(x)$, continuous, and not identically zero. Unless $K(x, y)$ is the sum of a finite number of products like $\pm\phi(x)\phi(y)$, there is an infinite number of such values of λ, and, in case it converges uniformly, the following development holds for $K(x, y)$:

$$K(x, y) = \frac{\phi_0(x)\phi_0(y)}{\lambda_0} + \frac{\phi_1(x)\phi_1(y)}{\lambda_1} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2} + \ldots, \tag{2}$$

where $\phi_i(x)$ is the normed solution of (1) corresponding to λ_i. The functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., form an orthogonal set on the interval $(0, 1)$. They will be called the "harmonics" of the kernel $K(x, y)$, and the corresponding values of λ_i, their "frequencies." The "iterated kernels" are defined by

$$K_j(x, y) = \int_0^1 K_{j-1}(x, r) K(r, y) dr, \text{ and } K_0(x, y) = K(x, y). \tag{3}$$

For these, the developments, known to be uniformly convergent[‡] for $i \geqq 2$, hold:

$$K_j(x, y) = \frac{\phi_0(x)\phi_0(y)}{\lambda_0^{j+i}} + \frac{\phi_1(x)\phi_1(y)}{\lambda_1^{j+i}} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2^{j+i}} + \ldots.. \tag{4}$$

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[*] Presented to the American Mathematical Society, September 4, 1917.

[†] See, for instance, Schmidt, "Entwickelung wilkürlicher Functionen nach Systemen vorgeschriebener," Diss. Göttingen, 1915; Bôcher, "An Introduction to the Study of Integral Equations," Cambridge tracts in math. and math. phys., 1909; Kowalewski, "Einführung in die Determinantentheorie," Veit u. Comp., Leipzig, 1909.

[‡] Kowalewski, *loc. cit.*, p. 533.

In a previous paper[*] I have shown that a number of the oscillation properties of the more common orthogonal sets are consequences of continuity, orthogonality, and the added condition:

(D): $D_n(x_0, x_1, \ldots, x_n) > 0$ on the range R: $0 < x_0 < x_1 \ldots x_n < 1$, for $n = 0, 1, 2, \ldots$, where

$$D_n(x_0, x_1, \ldots, x_n) = \begin{vmatrix} \phi_0(x_0), \phi_1(x_0), \ldots, \phi_n(x_0), \\ \phi_0(x_1), \phi_1(x_1), \ldots, \phi_n(x_1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \phi_0(x_n), \phi_1(x_n), \ldots, \phi_n(x_n), \end{vmatrix}$$

and $D_0(x_0) = \phi_0(x_0)$.

It was stated that it appeared desirable to connect this condition with the theory of integral equations. The following is intended as a contribution to this point.

2. *The Condition on $K(x, y)$.*

To find the necessary and sufficient condition on $K(x, y)$ in order that its harmonics may have the property (D) presents difficulties, since the same harmonics may arise from a variety of kernels, the frequencies being altered. However, a condition may be found which is simple, and which appears to be satisfied in the more common cases. We proceed to indicate considerations which suggest this condition, and which make it appear useful.

It is a property of the more common sets that if $\Phi_{0n}(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x)$ is a linear combination of the harmonics of $K(x, y)$, the function $\int_0^1 \Phi_{0n}(y) K(x, y)\, dy$ has no more sign changes in the interior of $(0, 1)$ than $\Phi_{0n}(x)$, and in many cases, this property holds for any continuous function $f(x)$. Assuming that it does, let $y_0, y_1, \ldots, y_n$ be any $n+1$ points on R. Choose for $f(x)$ any continuous function $f(x, \epsilon)$ equal to 0 except on the sub-intervals $(y_i - \epsilon, y_i + \epsilon)$, where it is determined so that

$$\int_{y_i - \epsilon}^{y_i + \epsilon} f(x, \epsilon)\, dx = (-1)^{i-1} c_i,$$

the c_i being independent of ϵ, and $f(x, \epsilon)$ not changing sign on any sub-interval. If the c_i are all of one sign, $f(x, \epsilon)$ will have n sign changes, otherwise fewer.

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[*] AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 1 (1916), p. 1. The condition (D) employed in that paper, and in this, which requires determinants of the functions $\phi_i(x)$ to be positive, is in reality no more restrictive than that they be different from zero, as is evident when we reflect that the function $\phi_n(x)$ may be replaced, if necessary, by its negative, without affecting any other hypothesis. But it is more convenient to retain it in the original form.

If now the law of the mean be applied to the integral $\int_0^1 f(y, \varepsilon) K(x, y)\,dy$, and the limit taken as $\varepsilon = 0$, the result is

$$f_1(x) = c_0 K(x, y_0) - c_1 K(x, y_1) + \ldots. + (-1)^{n-1} c_n K(x, y_n),$$

and it may be shown, on our assumption, that this has no more sign changes that $f(x, \varepsilon)$. In other words, there is no set of positive numbers $a_0, a_1, \ldots.,$ a_{n+1}, and no set of points $x_0, x_1, \ldots., x_{n+1}$ on R for which the following equations are possible:

$$\left.\begin{array}{r} c_0 K(x_0, y_0) - c_1 K(x_0, y_1) + \ldots. + (-1)^n c_n K(x_0, y_n) = a_0, \\ -[c_0 K(x_1, y_0) - c_1 K(x_1, y_1) + \ldots. + (-1)^n c_n K(x_1, y_n) = a_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ (-1)^{n+1}[c_0 K(x_{n+1}, y_0) - c_1 K(x_{n+1}, y_1) + \ldots. + (-1)^n c_n K(x_{n+1}, y_n) = a_{n+1}. \end{array}\right\} \tag{5}$$

Now let $M_0, M_1, \ldots., M_{n+1}$ be the determinants formed from the matrix

$$\|K(x_i, y_j)\|, \quad i = 0, 1, \ldots., n+1, \quad j = 0, 1, \ldots., n \tag{6}$$

by omitting the first, second, third rows, and so forth. No two of the quantities M_i can differ in sign, since in the contrary case the equation $M_0 a_0 + M_1 a_1 + \ldots. + M_{n+1} a_{n+1} = 0$ would be possible with all a_i positive, and for such a_i the equations (5) would be compatible. So we conclude that no two of the determinants

$$K\begin{pmatrix} x_0, & x_1, & \ldots., & x_n \\ y_0, & y_1, & \ldots., & y_n \end{pmatrix} = \begin{vmatrix} K(x_0, y_0), & K(x_0, y_1), & \ldots., & K(x_0, y_n) \\ K(x_1, y_0), & K(x_1, y_1), & \ldots., & K(x_1, y_n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ K(x_n, y_0), & K(x_n, y_1), & \ldots., & K(x_n, y_n) \end{vmatrix} \tag{7}$$

taken from the same matrix (6), can have opposite signs. If, now, we can pass through a chain of such determinants, different from 0, from any given one to any given second, we may conclude that the determinant (7) is either never negative, or never positive on the region *RR:* $0 < x_0 < x_1 \ldots. < x_n < 1$. $0 < y_0 < y_1 \ldots. < y_n < 1$. Thus is indicated the condition of the next paragraph,

3. *A Sufficient Condition Upon $K(x, y)$.*

We shall suppose the function $K(x, y)$, in addition to being real, continuous, and symmetric on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, to have the property:

(K) : $K\begin{pmatrix} x_0, & x_1, & \ldots., & x_n \\ y_0, & y_1, & \ldots., & y_n \end{pmatrix} \geqq 0$ on *RR* for $n = 0, 1, 2, \ldots.,$ the *inequality* holding when $x_i = y_i$ for all i. By $K\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is understood $K(x_0, y_0)$.

If this property is not enjoyed by $K(x, y)$, it may be by one of the iterated kernels (3), which has the same harmonics. We proceed to prove the theorem:

If $K(x, y)$ *is real, continuous, and symmetric on the square* $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, *and has the property* (K), *its harmonics have the property* (D), *with, of course, all the properties derived in the paper referred to.*

We begin with the following special case as a lemma:

If $K_0(x, y)$ *is real, continuous and symmetric and satisfies* (K) *for* $n=0$, *it has but one frequency of least absolute value, and but one harmonic corresponding to this frequency. This harmonic never vanishes on the interior of the interval* $(0, 1)$, *and its frequency is positive.*

It should be noticed that the property (K) for $n=0$ is retained in the iterated kernels. We start from the first iterated kernel, $K_1(x, y)$, whose frequencies, being the squares of those of $K(x, y)$ are all positive. Its least frequency is thus λ_0^2, which we may assume to be 1 without impairing the argument, since we may replace $K_1(x, y)$ by $\lambda_0^2 K_1(x, y)$, the latter kernel having 1 as least frequency. If more than one harmonic corresponded to 1, the development (2) would take the form:

$$K_1(x, y) = \phi_0(x)\phi_0(y) + \phi_1(x)\phi_1(y) + \dots + \phi_n(x)\phi_n(y) \\ + \frac{\phi_{n+1}(x)\phi_{n+1}(y)}{\lambda_{n+1}^2} + \frac{\phi_{n+2}(x)\phi_{n+2}(y)}{\lambda_{n+2}^2} + \dots, \tag{8}$$

where in the first line appear the harmonics corresponding to 1, and in the second, those corresponding to frequencies greater than 1. Repeated iteration diminishes these later terms, and in the limit we have a function

$$F(x, y) = \phi_0(x)\phi_0(y) + \phi_1(x)\phi_1(y) + \dots + \phi_n(x)\phi_n(y), \tag{9}$$

which is continuous, never negative, and symmetric. The harmonics composing it may be assumed to be normed and orthogonal. $F(x, y)$ may, moreover, be shown to be *positive throughout the whole interior of the square* $0 < x < 1$, $0 < y < 1$, as follows. It has positive elements, since it is not identically 0, and it is, for any fixed y, a solution of the integral equation $f(x) = \int_0^1 f(y) K_1(x, y)\,dy$. Let b be a value of y such that $f(x) = F(x, b)$ is positive at some point $0 < x < 1$. Then $f(x)$ can not be 0 at any interior point, for if it were, there would be an interior point, a, terminating an interval on which $f(x) > 0$, and this would lead to the contradiction

$$f(a) = 0 = \int_0^1 f(y) K_1(a, y)\,dy,$$

the integrand being positive near $y=a$, and never negative. Thus $F(x, y)>0$ at all points of the line $y=b$, and being symmetric, at all points of $x=b$. Hence, the argument being repeated with any other value of y, the conclusion is established. We shall now show that $F(x, y)$ being >0 within the square, the sum (9) reduces to a single product.

We must distinguish three cases according to the behavior of $F(x, y)$ on the sides of the square. We note that the equation holds:

$$F(x, y)=\int_0^1 F(x, r) F(r, y) dr. \tag{10}$$

If $F(x, y)$ vanishes at a boundary point, say $(0, b)$, we have

$$0=\int_0^1 F(0, r) F(r, b) dr,$$

and as the integrand is never negative, it must vanish for all r. If $0<b<1$, $F(r, b)>0$ for all interior r, and $F(0, r)=0$. If $b=0$, the integrand is a square, and the same conclusion follows. If $b=1$, either one of the factors must vanish on the whole open interval, or one factor must vanish at some points, and the other at all the rest at least. But the last alternative is impossible, since (10) will show that if $F(x, y)$ is positive at one boundary point, it will be positive on the whole open side. We conclude that if $F(x, y)$ vanishes at any boundary point between corners, it vanishes on the whole side corresponding, and by symmetry, on a second side. The cases, then, are: I, $F(x, y)>0$ on the closed square; II, $F(x, y)>0$ on the open square, but vanishing on the whole boundary; and III, $F(x, y)>0$ on the open square and on two symmetric sides, but vanishing on the other two.

If b is any interior point, $F(x, b)$ is a harmonic, since (9) shows it to be a linear homogeneous combination of harmonics with constant coefficients. It may then be normalized and considered one of the harmonics $\phi_i(x)$, since if the functions of (9) be subjected to an orthogonal transformation, its form remains unaltered, the harmonics remaining normed and orthogonal. We may therefore identify $\phi_0(x)$ with $F(x, b)/\sqrt{\int_0^1 F^2(x, b) dx}$.

CASE I. Here $\phi_0(x)>0$ on the closed interval $(0, 1)$. If there is a second function $\phi_1(x)$, its ratio to $\phi_0(x)$ is continuous on the closed interval, and hence attains its maximum, M, say for $x=a$. Then $M\phi_0(x)-\phi_1(x)\geqq 0$ on $(0, 1)$. But as

$$M\phi_0(x)-\phi_1(x)=\int_0^1 F(x, y)\left[M\phi_0(y)-\phi_1(y)\right] dy, \tag{11}$$

we have, for $x=a$, a contradiction, since the integrand is continuous, has positive elements, and is never negative, while the left-hand side vanishes.

Case II. Here, $F(x, y)$ being uniformly continuous, and vanishing on the whole boundary, it is possible, given $\varepsilon>0$, to find $\delta>0$, such that $0 \leqq F(x, y) < \delta$ for $0 \leqq x \leqq \delta$, and all y, and for $1-\delta \leqq x \leqq 1$ and all y, and for the corresponding regions formed by interchanging x and y. A similar condition holds for all harmonics as a consequence. For, for $0 \leqq x \leqq \delta$, or $1-\delta \leqq x \leqq 1$, $\phi_i(x) = \int_0^1 F(x, y)\phi_i(y)\,dy$ is less in absolute value than

$$\varepsilon \int_0^1 |\phi_i(y)|\,dy \leqq \varepsilon \sqrt{\int_0^1 \phi_i^2(y)\,dy} = \varepsilon.$$

Now, if there be a second harmonic, the ratio $\phi_1(x)/\phi_0(x)$ is continuous on the closed interval $(\delta, 1-\delta)$, and attains its maximum M, say for $x=a$ within this interval. Then $M\phi_0(x)-\phi_1(x) \geqq 0$ on $(\delta, 1-\delta)$, and on the rest of $(0, 1)$, since $\phi_0(x) \geqq 0$, $M\phi_0(x)-\phi_1(x) > -\varepsilon$. But $M\phi_0(x)-\phi_1(x) = \int_0^1 F(x, y)\,[M\phi_0(y)-\phi_1(y)]\,dy = \int_\delta^{1-\delta} + \int_0^\delta + \int_{1-\delta}^1$, and as the first integral is never negative, we conclude that for $0 \leqq x \leqq \delta$, or $1-\delta \leqq x \leqq 1$, $M\phi_0(x) - \phi_1(x) > -2\delta\varepsilon^2$. If the process is repeated, we see that this function is greater than $-(2\delta\varepsilon)^2\varepsilon$, and so on, so that the lower limit of $M\phi_0(x)-\phi_1(x)$ can only be 0 on the whole interval $(0, 1)$. We may then complete the reasoning as in Case I.

Case III. A combination of the methods employed in the first two cases leads to the same result, that there can not be a second harmonic.

Thus $\phi_0(x)$, except for a multiplicative constant, is the only harmonic of $K_1(x, y)$ of frequency 1, and therefore the only harmonic of $K(x, y)$ of frequency $+1$ or -1. We have seen that it does not vanish within $(0, 1)$. Since it satisfies the integral equation (1) its frequency must be $+1$, because the integral and the left member of the equation have the same signs. This completes the proof of the lemma.

It is important for what follows to notice that though the lemma is proven for a kernel in two variables, the proof can be easily adapted to a real, continuous kernel symmetric in two sets of variables.

The lemma has established the property (D) for the harmonics $\phi_i(x)$ for $n=0$. To extend the argument to all values of n, we proceed to derive integral equations which the determinants $D_1, D_2, \ldots.$ satisfy, and apply the lemma to their kernels.

An abbreviation will be helpful. Let ξ stand for the aggregate of variables $x_0, x_1, \ldots, x_n$; η for $y_0, y_1, \ldots, y_n$; and let us write $\varkappa(\xi, \eta)$ for $K\begin{pmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{pmatrix}$; and $\Delta(\xi)$ for $D_n(x_0, x_1, \ldots, x_n)$. Let S stand for the region $0 \leqq x_i \leqq 1$, $i = 0, 1, \ldots, n$, and by $\int_S f(\xi)d\xi$ let us understand

$$\int_0^1 \int_0^1 \cdots \int_0^1 f(x_0, x_1, \ldots, x_n)dx_0, dx_1, \ldots, dx_n.$$

4. *The Integral Equation for* $\Delta(\xi)$.

We apply to $\Delta(\eta)\varkappa(\xi, \eta)$ Lagrange's product formula:

$$\Delta(\xi)\varkappa(\xi, \eta) = \begin{vmatrix} \sum_0^n \phi_0(y_i)K(x_0, y_i), & \sum_0^n \phi_0(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_0(y_i)K(x_n, y_i) \\ \sum_0^n \phi_1(y_i)K(x_0, y_i), & \sum_0^n \phi_1(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_1(y_i)K(x_n, y_i) \\ \cdots & \cdots & \cdots & \cdots \\ \sum_0^n \phi_n(y_i)K(x_0, y_i), & \sum_0^n \phi_n(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_n(y_i)K(x_n, y_i) \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} \phi_0(y_{i_0})K(x_0, y_{i_0}), & \phi_0(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_0(y_{i_n})K(x_n, y_{i_n}) \\ \phi_1(y_{i_0})K(x_0, y_{i_0}), & \phi_1(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_1(y_{i_n})K(x_n, y_{i_n}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_n(y_{i_0})K(x_0, y_{i_0}), & \phi_n(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_n(y_{i_n})K(x_n, y_{i_n}) \end{vmatrix}$$

where in the summation $i_0, i_1, \ldots, i_n$ take on all values from 0 to n. However, if any two of the indices are equal, the corresponding determinant is 0, two columns becoming proportional, so that the indices may be restricted to the $(n+1)!$ permutations of the numbers $0, 1, \ldots, n$. Thus we have $(n+1)!$ terms, which, after integration over the field S, all become equal, namely to

$$\begin{vmatrix} \phi_0(x_0)/\lambda_0, & \phi_0(x_1)/\lambda_0, & \ldots, & \phi_0(x_n)/\lambda_0, \\ \phi_1(x_0)/\lambda_1, & \phi_1(x_1)/\lambda_1, & \ldots, & \phi_1(x_n)/\lambda_1, \\ \cdots & \cdots & \cdots & \cdots, \\ \phi_n(x_0)/\lambda_n, & \phi_n(x_1)/\lambda_n, & \ldots, & \phi_n(x_n)/\lambda_n. \end{vmatrix}$$

Hence the required integral equation for $\Delta(\xi)$ is

$$\chi(\xi) = \mu\int_0^1 \chi(\eta)\varkappa(\xi, \eta)d\eta, \tag{12}$$

the solution $\Delta(\xi)$ corresponding to $\mu = (\lambda_0\lambda_1 \ldots \lambda_n)/(n+1)!$.

Various other solutions are obtained by forming determinants of sets of $n+1$ of the functions $\phi_i(x)$, other than the first $n+1$, the frequencies being the products of the corresponding frequencies of the $\phi_i(x)$ divided by $(n+1)!$

In order to apply the lemma of the last paragraph, it is necessary to know that the frequency of $\Delta(\xi)$ is the one of smallest absolute value of the kernel $\varkappa(\xi, \eta)$. To do this, we shall show that $\varkappa(\xi, \eta)$ has no other frequencies than those of the form $=\pm(\lambda_{i_0}\lambda_{i_1}\ldots\lambda_{i_n})/(n+1)!$. This is the subject of the next paragraph.

5. *The Frequencies of $\varkappa(\xi, \eta)$.*

We shall here make use of the following fact: $\lambda_0, \lambda_1, \ldots,$ *being a set of real quantities, they are determined by the power sums* $s_{-1}=\Sigma 1/\lambda_i$, $s_{-2}=\Sigma 1/\lambda_i^2, \ldots,$ *in case these sums are absolutely convergent.* The truth of this fact follows from our knowledge that the power sums determine a transcendental integral function whose roots the λ_i are.* Let us now assume, the justification to follow later, that $K(x, y)$ is the first iterated kernel of a real continuous symmetric kernel. It will follow from (4) that the power sums all converge absolutely, and will be given by

$$s_{-j}=\int_0^1 K_{j-1}(r, r)\,dr=\sum_i 1/\lambda_i^j. \tag{13}$$

Under the assumption just made, $\varkappa(\xi, \eta)$ will also be the first iterated kernel of a real, symmetric, continuous kernel and the power sums of its frequencies, also absolutely convergent, will be given by

$$s'_{-j}=\int_S \varkappa_{j-1}(\rho, \rho)\,d\rho=\sum_i 1/\mu_i^j. \tag{14}$$

We are concerned with establishing the equations:

$$s'_{-j}=\sum_{i_0, i_1, \ldots, i_n}\left[\frac{(n+1)!}{\lambda_{i_0}\lambda_{i_1}\ldots\lambda_{i_n}}\right]^j, \qquad j=1, 2, \ldots. \tag{15}$$

In order to prove this formula for $j=1$, we introduce the notation p_n, defining it by the equation:

$$\begin{aligned}(n+1)!\,p_{n+1}&=\int_S K\begin{pmatrix} r_0, & r_1, & \ldots, & r_n \\ r_0, & r_1, & \ldots, & r_n\end{pmatrix} dr_0\,dr_1\ldots dr_n \\ &=\int_0^1\int_0^1\ldots\int_0^1 \begin{vmatrix} K(r_0, r_0), & K(r_0, r_1), & \ldots, & K(r_0, r_n) \\ K(r_1, r_0), & K(r_1, r_1), & \ldots, & K(r_1, r_n) \\ \ldots & \ldots & \ldots & \ldots \\ K(r_n, r_0), & K(r_n, r_1), & \ldots, & K(r_n, r_n)\end{vmatrix} dr_0\,dr_1\ldots dr_n,\end{aligned} \tag{16}$$

and shall prove that p_n is the sum of the products of the reciprocals of the λ_i n at a time. It will be noticed that the terms of the expanded determinant contain only products of the form $K(r_{i_1}, r_{i_2})K(r_{i_2}, r_{i_3})\ldots.K(r_{i_{k-1}}, r_{i_k})K(r_{i_k}, r_{i_1})$, which, upon integration becomes $\int_0^1 K_{k-1}(r, r)\,dr, = s_{-k}$. Fixing our attention upon those products containing a factor from the first row and one from the first column, we easily verify the following recursion formula for p_{n+1}:

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* Log $P=\log(1-z/\lambda_0)(1-z/\lambda_1)(1-z/\lambda_2)\ldots$ has these sums, apart from numerical factors, as coefficients, when expanded in a power series.

$$(n+1)p_{n+1} = s_{-1}p_n - s_{-2}p_{n-1} + s_{-3}p_{n-2} - \ldots. + (-1)^n s_{-1}.$$

But this is the recursion formula for the product sums in terms of the power sums given in the theory of algebraic equations, and applicable here because of the convergence of the power sums. And as (16) also gives $p_1 = s_{-1}$, it is clear that p_n is the product sum stated.*

On the other hand, $(n+1)!\,p_{n+1}$, by (16) is $\int_S \varkappa(\rho, \rho)\,d\rho = s'_{-1}$, so that (15) is proved for $j=1$.

To extend the proof is not difficult. We compute the value of the integral $\varkappa_1(\xi, \eta) = \int_S \varkappa(\xi, \rho)\varkappa(\rho, \eta)\,d\eta$, just as we did $\int_S \Delta(\eta)\varkappa(\xi, \eta)\,d\eta$ in the last paragraph, obtaining

$$\varkappa_1(\xi, \eta) = (n+1)!\begin{vmatrix} K_1(x_0, y_0), & K_1(x_0, y_1), & \ldots., & K_1(x_0, y_n) \\ K_1(x_1, y_0), & K_1(x_1, y_1), & \ldots., & K_1(x_1, y_n) \\ \cdots & \cdots & \cdots & \cdots \\ K_1(x_n, y_0), & K_1(x_n, y_1), & \ldots., & K_1(x_n, y_n) \end{vmatrix}, \qquad (17)$$

This shows that the power sum s'_{-2} may be computed just as was s'_{-1}, only replacing $K(x, y)$ by $K_1(x, y)$ and multiplying by $(n+1)!$. But the effect of replacing $K(x, y)$ by $K_1(x, y)$ is to replace the numbers λ_i by their squares. Thus the formula (15) is demonstrated for $j=2$. It is clear that the method is general, and we may regard (15) as true for all positive integers j.

Reverting to our assumption that $K(x, y)$ was the first iterated kernel of a continuous symmetric kernel, we note that should there be doubt of this, we may, in the above reasoning, begin with $K_1(x, y)$ instead of $K(x, y)$. Equation (17) then shows that $\varkappa(\xi, \eta)$ would be replaced by $\varkappa_1(\xi, \eta)/(n+1)!$, and instead of the frequencies of $\varkappa(\xi, \eta)$ being identified with the products $(\lambda_{i_0}\lambda_{i_1}\ldots.\lambda_{i_n})/(n+1)!$, we should only have their squares identified with the squares of these products. But this is sufficient for the purposes in hand, and gives the result announced at the close of the last paragraph.

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* The result has an interesting bearing on the Fredholm series for symmetric kernels, for p_n is the coefficient of $(-\lambda)^n$ in the series $d(\lambda) = 1 - \frac{\lambda}{1!}\int_0^1 K(r, r)\,dr + \frac{\lambda^2}{2!}\int_0^1\int_0^1 K\begin{pmatrix} r_0, & r_1 \\ r_0, & r_1 \end{pmatrix} dr_0\,dr_1 - \ldots.$, whose roots are the frequencies of $K(x, y)$. It shows that if the power sums are absolutely convergent, as they will be, for example, if $K(x, y)$ is the first iterated kernel of a real, symmetric, continuous kernel, the transcendental integral function $d(\lambda)$ *contains as factor no transcendental integral function without zeros.*

6. *The Property* (*D*).

Evidently the sign of $\Delta(\xi)$ is not constant on S, as an interchange of two arguments changes the sign of $\Delta(\xi)$. But we are concerned only with showing it different from 0 on R. We therefore substitute in the equation (12), the field R for the field S. In doing so, we note that $\Delta(\eta)$ and $\varkappa(\xi, \eta)$ are alternating functions on S for η, and that therefore their product is symmetric. Hence the field R is one of $(n+1)\,!$ symmetric sub-fields for the integral, corresponding to one of the $(n+1)\,!$ orders of the arguments y. Hence $\int_S \Delta(\eta)\varkappa(\xi, \eta)\,d\eta = (n+1)\,!\int_R \Delta(\eta)\varkappa(\xi, \eta)\,d\eta$, so that $\Delta(\xi)$ now satisfies the equation

$$\chi(\xi) = \nu \int_R \chi(\eta)\varkappa(\xi, \eta)\,d\eta \tag{18}$$

for $\nu = \lambda_0\lambda_1 \ldots . \lambda_n$. This equation will have the same harmonics as (12), and its frequencies will evidently be those of (12) multiplied by $(n+1)\,!$.

The hypothesis (K) on $K(x, y)$ includes the hypothesis (K) for $n=0$ on $\varkappa(\xi, \eta)$, and the lemma of paragraph 3 applies. As the frequency of least absolute value of $\varkappa(\xi, \eta)$ is $\pm\lambda_0\lambda_1 \ldots . \lambda_n$, since this is the smallest (or one of the smallest, in case several are equal) product of $n+1$ of the λ_i in absolute value, the single harmonic belonging to this frequency must be $\Delta(\xi)$. And this one harmonic does not vanish on R. Hence, by proper choice of the sign of $\phi_n(x)$, we may conclude $\Delta(\xi) = D_n(x_0, x_1, \ldots . , x_n) > 0$ on R, and the property (D) is thus generally established for the harmonics of $K(x, y)$.

7. *A Generalization.*

As, in the foregoing, we have used the hypothesis (K) only for one value of n, the following theorem follows:

If the real, continuous, symmetric kernel $K(x, y)$ *satisfies the hypothesis* (K) *for* $n = n_1, n_2, \ldots .$, *then the harmonics of* $K(x, y)$ *satisfy the condition* (D) *for* $n = n_1, n_2, \ldots . .$

In the next number of this Journal I shall establish the property (D) for the orthogonal function sets arising from ordinary linear homogeneous differential equations of the second order.

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